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ORLICZ NORM ESTIMATES FOR EIGENVALUES OF MATRICES

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ABSTRACT

Let φ be a supermultiplicative Orlicz function such that the function $t \mapsto \varphi(\sqrt{t})$ is equivalent to a convex function. Then each complex $n \times n$ matrix $T = (\tau_{ij})_{i,j}$ satisfies the following eigenvalue estimate:

$$\|(\lambda_i(T))_{i=1}^n\|_{\ell_\varphi} \leq C\|(\|(\tau_{ij})_{i=1}^n\|_{\ell_{\varphi_*}})_{j=1}^n\|_{\ell_{\overline{\varphi}}}.$$

Here, φ_* stands for Young's conjugate function of φ, $\overline{\varphi}$ is the minimal submultiplicative function dominating φ and $C > 0$ a constant depending only on φ. For the power function $\varphi(t) = t^p$, $p \geq 2$ this is a celebrated result of Johnson, König, Maurey and Retherford from 1979. In this paper we prove the above result within a more general theory of related estimates.

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1. Preliminaries

We use standard notation and notions from Banach space theory. For $1 \le p \le \infty$ its conjugate number p' is defined by the equality $1/p+1/p' = 1$. If X is a Banach space, then X' denotes its dual space, and for Banach spaces X and Y we mean by $T: X \to Y$ that T is a linear and continuous operator between these two spaces, with dual operator $T': Y' \to X'$; the collection of all such operators T is denoted by $L(X,Y)$. If we write $X \hookrightarrow Y$, then we assume that $X \subset Y$ and that the identity map id: $X \to Y$ is continuous. For an operator $T: X \to X$ on an n-dimensional Banach space X we denote by $\lambda_1(T), \dots, \lambda_n(T)$ the collection of its eigenvalues, counted according to their algebraic multiplicity and arranged in decreasing order of their absolute values, i.e., $|\lambda_1(T)| \ge |\lambda_2(T)| \ge \cdots \ge |\lambda_n(T)|$.

By a Banach sequence space we mean a complex Banach lattice E modelled on the set of natural numbers $\mathbb{N}$ which contains a sequence x with $\operatorname{supp} x = \mathbb{N}$. E is said to be symmetric provided that $\|x\|_E = \|x^*\|_E$ for all $x \in E$, where x^* denotes the decreasing rearrangement of x. The n-th standard unit vector in E is denoted by e_n and the linear span of the first n standard unit vectors (equipped with the induced norm) by E_n.

For the theory of Orlicz functions and Orlicz spaces we refer to [16]. Two functions $f, g: [0,\infty) \to [0,\infty)$ are called equivalent whenever there exist constants $a,b,c,d > 0$ such that $f(t) \le ag(bt) \le cf(dt)$ for all $t \in [0,\infty)$. A continuous and convex function $\varphi: [0,\infty) \to [0,\infty)$ with $\varphi^{-1}(\{0\}) = \{0\}$ is called an Orlicz function. It is said to be submultiplicative, respectively, supermultiplicative, if there exists a constant $C > 0$ such that for all $s,t > 0$ the inequality $\varphi(st) \le C\varphi(s)\varphi(t)$, respectively, $\varphi(st) \ge C\varphi(s)\varphi(t)$, holds. An Orlicz function φ satisfies the Δ_2-condition, written $\varphi \in \Delta_2$, if $\sup_{t\ge 0} \varphi(2t)/\varphi(t) < \infty$. The function $\varphi_*: [0,\infty) \to [0,\infty]$ defined by

$$\varphi_*(t) := \sup_{s\ge 0}(st - \varphi(s))$$

is Young's conjugate function of φ which takes finite values provided that $\lim_{t\to\infty} \varphi(t)/t = \infty$.

Recall that for an Orlicz function φ the vector space

$$\ell_\varphi := \{x \in \mathbb{C}^{\mathbb{N}}; \sum_{n=1}^{\infty} \varphi(|x_n|/\varepsilon) < \infty \text{ for some } \varepsilon > 0\}$$

together with the Minkowski functional

$$\|x\|_{\ell_\varphi} := \inf\{\varepsilon > 0; \sum_{n=1}^{\infty} \varphi(|x_n|/\varepsilon) \le 1\}$$

forms a symmetric Banach sequence space—the Orlicz sequence space associated with φ. It is an easy excercise to see that equivalent Orlicz functions give the same Orlicz spaces (with equivalent norms).

2. General estimates

Let E, F and G be Banach sequence spaces and $T = (\tau_{ij})$ be an $n \times n$ matrix. Our general aim is to give upper estimates of

$$\Lambda_E(T) := \Big\| \sum_{i=1}^{n} \lambda_i(T) e_i \Big\|_E$$

in terms of

$$\|T\|_{F[G]} := \Big\| \sum_{j=1}^{n} \Big\| \sum_{i=1}^{n} \tau_{ij} e_i \Big\|_G e_j \Big\|_F .$$

To start, we need two further definitions. Let X be a Banach lattice and E be a Banach sequence space such that $\ell_2 \hookrightarrow E$. Then X is said to be $(E,2)$-concave, if there exists a constant $C > 0$ such that for arbitrarily many $x_1, \ldots, x_n \in X$

$$\Big\| \sum_{k=1}^{n} \|x_k\|_X e_k \Big\|_E \leq C \Big\| \Big(\sum_{k=1}^{n} |x_k|^2 \Big)^{1/2} \Big\|_X . \tag{1}$$

For $E = \ell_2$ this gives the well-known notion of 2-concavity, and more generally, for $E = \ell_q$, $2 \leq q < \infty$ the notion of cotype q as we will see later on.

For a Banach sequence space E such that $\ell_2 \hookrightarrow E$ an operator $T\colon X \to Y$ between Banach spaces is called $(E,2)$-summing, if there exists a constant $C > 0$ such that for finitely many and arbitrary $x_1, \ldots, x_n \in X$ the following inequality holds:

$$\Big\| \sum_{k=1}^{n} \|Tx_k\|_Y e_k \Big\|_E \leq C \sup_{\|x'\|_{X'} \leq 1} \Big(\sum_{k=1}^{n} |x'(x_k)|^2 \Big)^{1/2} . \tag{2}$$

In this case we denote by $\pi_{E,2}(T)$ the smallest constant $C > 0$ satisfying (2). The particular case $E = \ell_r$, $r \geq 2$ gives the classical notion of absolutely $(r,2)$-summing operators (cf. [7, Chapter 10]).

PROPOSITION 1: *Let E and F be Banach sequence spaces such that $\ell_2 \hookrightarrow F$ and E is $(F,2)$-concave. Then there exists $C > 0$ such that for all Banach spaces X and all $T\colon X \to E_n$*

$$\pi_{F,2}(T\colon X \to E_n) \leq C \Big\| \sum_{i=1}^{n} \|T' e_i\|_{X'} e_i \Big\|_E . \tag{3}$$

Proof: Take $x_1, \dots, x_m \in X$. Then

$$\|Tx_i\|_E = \|\sum_{k=1}^n T'e_k(x_i)e_k\|_E = \|\sum_{k=1}^n x'_k(x_i)e_k\|_E,$$

where $x'_k := T'e_k \in X'$. For each k for which $x'_k \neq 0$, set $z'_k := x'_k/\|x'_k\|_{X'} \in X'$ and $z'_k := 0$ otherwise, and let $y_i := \sum_{k=1}^n \|x'_k\|_{X'} z'_k(x_i)e_k \in E$. By the $(F,2)$-concavity of E there exists $C > 0$ such that

$$\begin{aligned}\Big\|\sum_{i=1}^m \|Tx_i\|_E e_i\Big\|_F &= \Big\|\sum_{i=1}^m \|y_i\|_E e_i\Big\|_F \\ &\leq C \cdot \Big\|\Big(\sum_{i=1}^m |y_i|^2\Big)^{1/2}\Big\|_E \\ &= C \cdot \Big\|\sum_{k=1}^n \|x'_k\|_{X'}\Big(\sum_{i=1}^m |z'_k(x_i)|^2\Big)^{1/2} e_k\Big\|_E \\ &\leq C \cdot \sup_{\|x'\|_{X'}\leq 1}\Big(\sum_{i=1}^m |x'(x_i)|^2\Big)^{1/2} \cdot \Big\|\sum_{k=1}^n \|x'_k\|_{X'} e_k\Big\|_E,\end{aligned}$$

which gives (3). ∎

To proceed we need the following result from [5] which generalizes an inequality due to König. For the sake of completeness we indicate a short proof. Recall that for an operator $T: X \to Y$ between Banach spaces the n-th approximation number $a_n(T)$ is defined by

$$a_n(T) := \inf\{\|T - T_n\|; T_n: X \to Y \text{ has rank} < n\},$$

and the n-th Weyl-number $x_n(T)$ by

$$x_n(T) := \sup\{a_n(TS); \|S: \ell_2 \to X\| \leq 1\}.$$

PROPOSITION 2: *Let E be a Banach sequence space such that $\ell_2 \hookrightarrow E$. Then for every $(E,2)$-summing operator T and all n*

$$\lambda_E(n)x_n(T) \leq \pi_{E,2}(T), \tag{4}$$

where $\lambda_E(n) := \|\sum_{k=1}^n e_k\|_E$.

Proof: Consider first the case where the operator T is defined on ℓ_2, say $T: \ell_2 \to Y$ for some Banach space Y. The proof of [15, 2.a.3] or [19, 2.7.1] yields that for $\varepsilon > 0$ there exists an orthonormal system (f_k) in ℓ_2 such that $a_k(T) \leq (1+\varepsilon)\|Tf_k\|_Y$ for all k (a result due to Pisier). This implies

$$\Big\|\sum_{k=1}^{n} a_k(T)e_k\Big\|_E \leq \pi_{E,2}(T),$$

since by Bessel's inequality $\sup_{\|x'\|_{\ell_2}\leq 1}\sum_{k=1}^{n}|(x'|f_k)|^2 \leq 1$. Now if for an arbitrary $(E,2)$-summing operator $T\colon X \to Y$ and $\varepsilon > 0$ we choose $S\colon \ell_2 \to X$ with $\|S\| = 1$ and $x_n(T) \leq (1+\varepsilon)a_n(TS)$, then by the monotonicity of the approximation numbers

$$\begin{aligned}(1+\varepsilon)^{-1}\lambda_E(n)x_n(T) \leq \lambda_E(n)a_n(TS) &\leq \Big\|\sum_{k=1}^{n} a_k(TS)e_k\Big\|_E \\ &\leq \pi_{E,2}(TS) \leq \pi_{E,2}(T),\end{aligned}$$

which gives the claim. ∎

The following crucial inequality connecting eigenvalues and Weyl-numbers is due to Weyl (for $E = \ell_p$) and König (general case, see [15, 2.a.8]). For simplicity we state a finite-dimensional version only. Let E be a symmetric Banach sequence space and $T\colon X \to X$ an operator on an n-dimensional Banach space X. Then

$$\Lambda_E(T) \leq 2\sqrt{2e}\Big\|\sum_{k=1}^{n} x_k(T)e_k\Big\|_E. \tag{5}$$

Combining (3), (4) and (5), we immediately obtain the following general estimate (by considering a matrix $T = (\tau_{ij})$ as an operator $T\colon E_n \to E_n$):

Proposition 3: *Let E and F be Banach sequence spaces such that F is symmetric with $\ell_2 \hookrightarrow F$ and E is $(F,2)$-concave. Then there exists a constant $C > 0$ such that for all n and every complex $n \times n$ matrix $T = (\tau_{ij})$*

$$\Lambda_F(T) \leq C\Big\|\sum_{i=1}^{n} \frac{1}{\lambda_F(i)}e_i\Big\|_F\|T\|_{E[E']}. \tag{6}$$

The use of Marcinkiewicz sequence spaces enables us to give upper bounds for single eigenvalues.

Proposition 4: *Let E and F be Banach sequence spaces such that F is symmetric with $\ell_2 \hookrightarrow F$ and E is $(F,2)$-concave. Then there exists a constant $C > 0$ such that for all n, every complex $n\times n$ matrix $T = (\tau_{ij})$ and all $1 \leq k \leq n$*

$$|\lambda_k(T)| \leq C\frac{1}{k}\sum_{i=1}^{k} \frac{1}{\lambda_F(i)}\|T\|_{E[E']}. \tag{7}$$

Proof: For an increasing sequence (λ_i) of positive scalars the symmetric Marcinkiewicz sequence space m_λ is the space of all scalar sequences $x = (x_i)$ for which $\|x\|_{m_\lambda} := \sup_{i\geq 1} x_i^{**}\lambda_i < \infty$, where $x_i^{**} := \frac{1}{i}\sum_{j=1}^{i} x_j^*$ (note that x_i^{**} is decreasing). In our case now set $\lambda_i := 1/(1/\lambda_F(i))^{**}$. Then (3) and (4) imply for all i and some $C > 0$

$$x_i^{**}(T\colon E_n \to E_n)\lambda_i \leq C\|T\|_{E[E']},$$

which by (5) gives

$$\lambda_k^{**}(T)\lambda_k \leq C\|T\|_{E[E']},$$

and (4) follows by $|\lambda_k(T)| = \lambda_k^*(T) \leq \lambda_k^{**}(T)$. ∎

3. Estimates for cotype q spaces

To fill (6) and (4) with a little life, we first consider spaces of cotype q. A Banach space X is said to be of cotype q, $2 \leq q < \infty$ if there exists a constant $C > 0$ such that for arbitrarily many $x_1, \dots, x_n \in X$

$$\left(\sum_{k=1}^{n} \|x_k\|^q\right)^{1/q} \leq C\left(\int_0^1 \Big\|\sum_{k=1}^{n} r_k(t)x_k\Big\|_X^2 dt\right)^{1/2}, \tag{8}$$

where as usual r_k denotes the k-th Rademacher function on $[0, 1]$. It is well-known that ℓ_p, $1 \leq p < \infty$ is of cotype $\max(2, p)$, and for further examples and references we refer to [7, Chapter 11].

LEMMA 5: *Let X be a Banach lattice and $2 \leq q < \infty$. Then X is of cotype q if and only if X is $(\ell_q, 2)$-concave.*

Proof: This is an immediate consequence of Maurey's generalization of the Khintchine inequality (cf. [17, 1.d.6]) together with [17, 1.f.9]. ∎

PROPOSITION 6: *Let E be a Banach sequence space of cotype q, $2 \leq q < \infty$. Then there exists a constant $C > 0$ such that for all n and every complex $n \times n$ matrix $T = (\tau_{ij})$,*

$$\Lambda_{\ell_q}(T) \leq C\log(1+n)^{1/q}\|T\|_{E[E']}, \tag{9}$$

and for all $1 \leq k \leq n$,

$$|\lambda_k(T)|k^{1/q} \leq C\|T\|_{E[E']}. \tag{10}$$

Proof: (9) follows from (6), the preceding lemma and the easy calculation $\|\sum_{i=1}^{n} 1/\lambda_{\ell_q}(i)e_i\|_{\ell_q} \leq \log(1+n)^{1/q}$. (10) follows from (7), the preceding lemma and the easy calculation $\frac{1}{k}\sum_{i=1}^{k} i^{-1/q} \leq c \cdot k^{-1/q}$ for some $c > 0$. ∎

4. Estimates for uniformly convex spaces

As we will see for Orlicz sequence spaces later on, the notion of cotype is often not optimal for our estimates. For a real Banach space X, the modulus of convexity $\delta_X(\varepsilon)$, $0 < \varepsilon \leq 2$ of X is defined by

$$\delta_X(\varepsilon) := \inf\{1 - \|x+y\|/2; x, y \in X, \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\}.$$

X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$. In this case, the modulus of convexity δ_X is equivalent (on $(0,2]$) to a canonic Orlicz function $\widetilde{\delta}_X$ (cf. [17, pp. 65ff]), and $\ell_2 \hookrightarrow \ell_{\widetilde{\delta}_X}$.

It is known (see [17, p. 78] or [2, pp. 310–311]) that if a Banach space X has modulus convexity of power type q, for some $q \geq 2$ (resp., modulus of smoothness of power type p, for some $1 < p \leq 2$), i.e., $\delta_X(\varepsilon) \geq C\varepsilon^q$ (resp., $\rho(\tau) := \sup_{0<\varepsilon\leq 2} \tau\varepsilon/2 - \delta_{X'}(\varepsilon) \leq C\tau^p$), then X is of cotype q (resp., is of type p). Furthermore, Pisier (see [20] or [2, pp. 273–296]) proved that every super-reflexive Banach space admits two equivalent norms, one which yields a space with modulus of convexity of power type q, for some $q \geq 2$, and one which yields a space with modulus of smoothness of power type p, for some $1 < p \leq 2$. This implies that every super-reflexive, and in particular every uniformly convex space, is of type p, for some $p > 1$, and cotype q, for some $q < \infty$. Hence, by the well-known fact (see [17, p. 92]) that every Banach lattice, which is of type p for some $p > 1$, is q-concave for some $q < \infty$, we conclude that every uniformly convex Banach lattice has finite concavity.

The content of the following lemma is essentially due to Figiel and Pisier [10], but we nevertheless state a proof for the convenience of the reader. One may use similar ideas to obtain an analogue for complex Banach spaces and the complex modulus of convexity using the results of [8] and [4]. We leave the details to the interested reader.

LEMMA 7: *Let X be a uniformly convex Banach space. Then X is of $\widetilde{\delta}_X$-cotype, i.e., there exists $C > 0$ such that for arbitrarily many $x_1, \dots, x_n \in X$*

$$\Big\|\sum_{k=1}^{n} \|x_k\|_X e_k\Big\|_{\ell_{\widetilde{\delta}_X}} \leq C\Big(\int_0^1 \Big\|\sum_{k=1}^{n} r_k(t)x_k\Big\|_X^2 dt\Big)^{1/2}. \tag{11}$$

Moreover, if X is a uniformly convex Banach lattice, then X is $(\ell_{\widetilde{\delta}_X}, 2)$-concave.

Proof: Denote by $L_2(X)$ the vector-valued L_2-space with respect to $[0,1]$, the Lebesgue-measure and X. Then as in the proof of [17, 1.e.16] (based on a result of Kadec [13]) we conclude that for all $x_1, \dots, x_n$ in X such that

$$\Big(\int_0^1 \|\sum_{k=1}^n r_k(t) x_k\|_X^2\Big)^{1/2} \leq 2$$

one has $\sum_{k=1}^n \delta_{L_2(X)}(\|x_k\|_X) \leq 1$. Since $\delta_{L_2(X)}$ is equivalent to δ_X (see e.g. [17, 1.e.9]) and therefore equivalent to $\widetilde{\delta}_X$, this implies that there exists a constant $K > 0$ such that for all $x_1, \dots, x_n$ in X with $(\int_0^1 \|\sum_{k=1}^n r_k(t)x_k\|_X^2)^{1/2} \leq K$ one has $\sum_{k=1}^n \widetilde{\delta}_X(\|x_k\|_X) \leq 1$ which implies (11). The second statement follows from the fact (see above) that every uniformly convex Banach lattice has finite concavity, together with Maurey's generalization of the Khintchine inequality. ∎

With these tools we are able to prove the following estimate:

PROPOSITION 8: *Let E be a uniformly convex Banach sequence space. Then there exists a constant $C > 0$ such that for all n and every complex $n \times n$ matrix $T = (\tau_{ij})$,*

$$\Lambda_{\ell_{\widetilde{\delta}_E}}(T) \leq C \log(1+n) \|T\|_{E[E']}, \tag{12}$$

where $\widetilde{\delta}_E$ is some Orlicz function equivalent to the modulus of convexity δ_E of E.

Proof: Since by Lemma 7 the lattice E is $(\ell_{\widetilde{\delta}_E}, 2)$-concave, the general estimate (6) leaves us with proving that for every Orlicz function φ we have $\|\sum_{i=1}^n 1/\lambda_{\ell_\varphi}(i) e_i\|_{\ell_\varphi} \leq \log(1+n)$. Without loss of generality we may assume that φ is strictly increasing. Then an easy calculation shows $\lambda_{\ell_\varphi}(i) = 1/\varphi^{-1}(1/i)$. Since for $\varepsilon = 1 + \log n$ we have $\varepsilon \geq 1$ for every n, it follows by the convexity of φ that

$$\sum_{i=1}^n \varphi(\varphi^{-1}(1/i)/\varepsilon) \leq \frac{1}{\varepsilon} \sum_{i=1}^n \frac{1}{i} \leq 1.$$

This implies by the definition of the Orlicz norm that

$$\Big\| \sum_{i=1}^n \varphi^{-1}(1/i) e_i \Big\|_{\ell_\varphi} \leq 1 + \log n,$$

which completes the proof. ∎

Since the modulus of convexity may vary heavily under renorming whereas the right-hand side in (12) does not (up to a constant, independent of the dimension and the matrix), one may improve the above estimate for a given Banach sequence space by suitable renormings.

5. Estimates for Lorentz and Orlicz norms

As a first concrete example we consider Lorentz sequence spaces. Let $1 \le p < \infty$ and let (w_n) be a non-increasing sequence of positive numbers such that $w_1 = 1$, $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$. The symmetric Banach sequence space of all sequences of scalars $x = (x_n)$ for which $\|x\| := \sup_\pi (\sum_{n=1}^{\infty} |x_{\pi(n)}|^p w_n)^{1/p} < \infty$, where π ranges over all the permutations of integers, is denoted by $d(w,p)$ and it is called a Lorentz sequence space.

PROPOSITION 9: *Let $d(w,p)$ be a Lorentz space. Assume that $2 \le p < \infty$ and the sequence (w_n) is such that $S(kn \ge cS(k)S(n))$ for some $c > 0$ and all k, n, where $S(n) = \sum_{i=1}^{n} w_i$. Then there exists $C > 0$ such that for all n and every complex $n \times n$ matrix $T = (\tau_{ij})$*

$$\Lambda_{\ell_{\varphi_S}}(T) \le C \log(1+n) \|T\|_{d(w,p)[d(w,p)']},$$

where $S(t)$ is a strictly increasing function on $[0,\infty)$ coinciding with $S(n)$ for $t = n$ and $\varphi_S(\varepsilon)$ an Orlicz function equivalent to the function $1/S^{-1}(1/\varepsilon^p)$.

Proof: Altshuler in [1] proved that $d(w,p)$ under the regularity assumption is uniformly convex and that its modulus of convexity is equivalent to the function $1/S^{-1}(1/\varepsilon^p)$, hence the result follows immediately from (12). ∎

Known estimates for the moduli of convexity of Orlicz sequence spaces allow us to prove the following result:

PROPOSITION 10: *Let φ be an Orlicz function such that the function $f(t) = \varphi(t)/t^2$, $t > 0$ is almost non-decreasing, i.e., there exists a constant $K > 0$ such that $f(s) \le Kf(t)$ provided $s \le t$.*

(i) *If φ is supermultiplicative, then there exists a constant $C > 0$ such that for all n and every complex $n \times n$ matrix $T = (\tau_{ij})$*

$$\Lambda_{\ell_\varphi}(T) \le C \log(1+n) \|T\|_{\ell_\varphi[\ell_{\varphi^*}]}. \tag{13}$$

(ii) *If φ is submultiplicative, then there exists a constant $C > 0$ such that for all n and every complex $n \times n$ matrix $T = (\tau_{ij})$*

$$\Lambda_{\ell_{\widetilde{\varphi}}}(T) \le C \log(1+n) \|T\|_{\ell_\varphi[\ell_{\varphi_*}]}, \tag{14}$$

where $\widetilde{\varphi}$ is some Orlicz function equivalent to the function $g(t) = 1/\varphi(1/t)$.

Proof: Again we deduce the desired statements from (12). The general condition on φ implies that $\varphi(t)/t \to \infty$ as $t \to \infty$. Thus φ_* takes finite values, and by the supermultiplicativity, respectively, submultiplicativity we have that $\varphi, \varphi_* \in \Delta_2$. This implies that $\ell'_\varphi = \ell_{\varphi_*}$, hence $\|T'e_i\|_{\ell'_\varphi} = \|\sum_{j=1}^n \tau_{ij} e_j\|_{\ell_{\varphi_*}}$. Furthermore, it was shown in [9, Corollary 22] and [18] that ℓ_φ admits an equivalent renorming such that for some $a, b, c > 0$ one has $\delta_{\ell_\varphi}(t) \ge a\psi(bt)$ on $[0, c]$, where

$$\psi(t) = \inf_{t \le u \le 1 \le v^{-1}} \frac{t^2}{u^2} \frac{\varphi(uv)}{\varphi(v)}.$$

The general condition on φ yields that there exists $d > 0$ such that $\psi(t) \ge d\varphi(t)$ if φ is supermultiplicative, which implies (i), and $\psi(t) \ge dg(t)$ if φ is submultiplicative. The general condition on φ also gives, similar to above, that $g(t)/t \to \infty$ as $t \to \infty$ which implies that g is equivalent to some Orlicz function $\widetilde{\varphi}$ which yields (ii). ∎

Note that the condition "$f(t) = \varphi(t)/t^2$ is almost non-decreasing" is equivalent to the condition "$f(t) = \varphi(\sqrt{t})$ is equivalent to a convex function in a neighborhood of zero" which is equivalent to ℓ_φ being 2-convex. Hence, our assumptions imply that in both cases ℓ_φ is of type 2 (cf. [17, 1.f.17]).

6. Orlicz norm estimates with no logarithmic terms

In order to obtain estimates with no logarithmic terms, a certain product trick is needed. For a scalar sequence $x = (x_1, \dots, x_n, 0, 0, \dots)$ with only finitely many non-zero entries we define

$$x \otimes x := (x_1x_1, \dots, x_1x_n, \dots, x_nx_1, \dots, x_nx_n, 0, 0, \dots),$$

and for a complex $n \times n$ matrix $T = (\tau_{ij})$ we denote its Kronecker/tensor product (see [11, 4.2.1]) by

$$T \otimes T := \begin{pmatrix} \tau_{11}T & \dots & \tau_{1n}T \\ \vdots & & \vdots \\ \tau_{n1}T & \dots & \tau_{nn}T \end{pmatrix}.$$

For our purposes we need a generalization of the well-known notions of upper and lower p-estimates (cf. [17, pp. 82–84]). Let E be a Banach sequence space and X a Banach lattice. X is said to satisfy an upper, respectively, lower E-estimate if there exists $C > 0$ such that for arbitrarily many $x_1, \dots, x_n \in X$

$$\Big\| \sup_{1\le k\le n} |x_k| \Big\|_X \le C \cdot \Big\| \sum_{k=1}^{n} \|x_k\|_X e_k \Big\|_E, \tag{15}$$

respectively,

$$\Big\| \sum_{k=1}^{n} \|x_k\|_X e_k \Big\|_E \le C \cdot \Big\| \sum_{k=1}^{n} |x_k| \Big\|_X. \tag{16}$$

If E itself satisfies an upper, respectively, lower E-estimate, then we denote by u_E, respectively, l_E, the infimum over all $C > 0$ satisfying (15), respectively, (16).

LEMMA 11: *Let E and F be symmetric Banach sequence spaces and $T = (\tau_{ij})$ be a complex $n \times n$ matrix.*

(i) *If E satisfies a lower E-estimate, then*

$$\Lambda_E(T)^2 \le l_E \Lambda_E(T \otimes T). \tag{17}$$

(ii) *If E satisfies an upper E-estimate and F an upper F-estimate, then*

$$\|T \otimes T\|_{E[F]} \le u_E u_F \|T\|^2_{E[F]}. \tag{18}$$

Proof: (i) It is easy to see that the set of eigenvalues of $T \otimes T$ equals $\{\lambda_i(T)\lambda_j(T); 1 \le i,j \le n\}$ and that the multiplicity of each of these products is the product of the multiplicities of $\lambda_i(T)$ and $\lambda_j(T)$, respectively (see e.g. [11, 4.2.12]), hence it is enough to show that $\|x\|_E^2 \le l_E \|x \otimes x\|_E$ whenever x is a finite sequence. Let $x = \sum_{i=1}^n x_i e_i$; then

$$x \otimes x = \sum_{j=1}^{n} \sum_{i=1}^{n} x_i x_j e_{i+(j-1)n} = \sum_{j=1}^{n} y_j,$$

where $y_j := x_j \sum_{i=1}^n x_i e_{i+(j-1)n}$ are pairwise disjoint. Hence, by the lower E-estimate and the symmetry of the norm in E

$$\begin{aligned} \|x \otimes x\|_E = \Big\| \sum_{j=1}^{n} |y_j| \Big\|_E &\ge l_E^{-1} \Big\| \sum_{j=1}^{n} x_j \Big\| \sum_{i=1}^{n} x_i e_{i+(j-1)n} \Big\|_E e_j \Big\|_E \\ &= l_E^{-1} \Big\| \sum_{i=1}^{n} x_i e_i \Big\|_E^2. \end{aligned}$$

(ii) It is $(T\otimes T)e_{j+(i-1)n}=\sum_{k=1}^{n}\sum_{\ell=1}^{n}\tau_{ki}\tau_{\ell j}e_{\ell+(k-1)n}$, hence, by the upper F-estimate and the symmetry of the norm in F,

$$\begin{aligned}\|(T\otimes T)e_{j+(i-1)n}\|_F &= \Big\|\sum_{k=1}^{n}\Big(\tau_{ki}\sum_{\ell=1}^{n}\tau_{\ell j}e_{\ell+(k-1)n}\Big)\Big\|_F\\ &= \Big\|\sup_{1\le k\le n}|\tau_{ki}|\Big\|\sum_{\ell=1}^{n}\tau_{\ell j}e_{\ell+(k-1)n}\Big\|\Big\|_F\\ &\le u_F\Big\|\sum_{k=1}^{n}|\tau_{ki}|\cdot\Big\|\sum_{\ell=1}^{n}\tau_{\ell j}e_{\ell+(k-1)n}\Big\|_F e_k\Big\|_F\\ &= u_F\Big\|\sum_{k=1}^{n}\tau_{ki}e_k\Big\|_F\Big\|\sum_{\ell=1}^{n}\tau_{\ell j}e_\ell\Big\|_F.\end{aligned}$$

With this we obtain, by the upper E-estimate and the symmetry of the norm in E,

$$\begin{aligned}&\Big\|\sum_{i=1}^{n}\sum_{j=1}^{n}\|(T\otimes T)e_{j+(i-1)n}\|_F e_{j+(i-1)n}\Big\|_E\\ &\qquad\le u_F\Big\|\sum_{i=1}^{n}\Big(\Big\|\sum_{k=1}^{n}\tau_{ki}e_k\Big\|_F\sum_{j=1}^{n}\Big\|\sum_{\ell=1}^{n}\tau_{\ell j}e_\ell\Big\|_F e_{j+(i-1)n}\Big)\Big\|_E\\ &\qquad= u_F\Big\|\sup_{1\le i\le n}\Big\|\sum_{k=1}^{n}\tau_{ki}e_k\Big\|_F\Big|\sum_{j=1}^{n}\Big\|\sum_{\ell=1}^{n}\tau_{\ell j}e_\ell\Big\|_F e_{j+(i-1)n}\Big|\Big\|_E\\ &\qquad\le u_E u_F\Big\|\sum_{j=1}^{n}\Big\|\sum_{k=1}^{n}\tau_{kj}e_k\Big\|_F e_j\Big\|_E^2,\end{aligned}$$

and obviously, the last term is equal to the right side of (18). ∎

This now enables us to prove our main result as announced in the abstract. For an Orlicz function φ let g: $[0,\infty)\to[0,\infty)$ be defined by $g(0)=0$ and $g(t)=1/\varphi(1/t)$, $t>0$. Under the assumptions of the forthcoming theorems, g is equivalent to some Orlicz function, which shall be denoted by $\widetilde{\varphi}$ henceforth. Note that in this case and if φ is supermultiplicative, $\widetilde{\varphi}$ is equivalent to the function $\overline{\varphi}$ from the abstract, the minimal submultiplicative function dominating φ defined by $\overline{\varphi}(t):=\sup_{s>0}\varphi(st)/\varphi(s)$, $t\ge 0$.

THEOREM 12: *Let φ be an Orlicz function such that the function $f(t)=\varphi(t)/t^2$ is almost non-decreasing.*

(i) *If φ is supermultiplicative, then there exists a constant $C>0$ such that for* *all n and every $n \times n$ matrix $T = (\tau_{ij})$*

$$\Lambda_{\ell_\varphi}(T) \le C\|T\|_{\ell_{\widetilde{\varphi}}[\ell_{\varphi_*}]}. \tag{19}$$

(ii) *If φ is submultiplicative, then there exists a constant $C > 0$ such that for all n and every $n \times n$ matrix $T = (\tau_{ij})$*

$$\Lambda_{\ell_{\widetilde{\varphi}}}(T) \le C\|T\|_{\ell_\varphi[\ell_{(\widetilde{\varphi})_*}]}. \tag{20}$$

Proof: (i) Since $\|\cdot\|_{\ell_\varphi} \le D\|\cdot\|_{\ell_{\widetilde{\varphi}}}$ for some constant $D > 0$, we obtain by (13)

$$\Lambda_{\ell_\varphi}(T) \le C \cdot \log(1+n) \cdot \|T\|_{\ell_{\widetilde{\varphi}}[\ell_{\varphi_*}]} \tag{21}$$

for some constant $C > 0$ independent of $n \in \mathbb{N}$ and $T \in \mathbb{C}^{n\times n}$. By Lemma 7 and what was said in the proof of Proposition 10 we conclude that ℓ_φ is $(\ell_\varphi, 2)$-concave and therefore satisfies a lower ℓ_φ-estimate, which implies by duality that ℓ_{φ_*} satisfies an upper ℓ_{φ_*}-estimate. Since $\widetilde{\varphi}$ is equivalent to a submultiplicative function, $\ell_{\widetilde{\varphi}}$ satisfies an upper $\ell_{\widetilde{\varphi}}$-estimate (cf. [6]). Together with (21) (for $T\otimes T$ instead of T) and Lemma 11 this gives

$$\Lambda_{\ell_\varphi}(T) \le 2(C\log(1+n))^{1/2} \cdot l_{\ell_\varphi} u_{\ell_{\widetilde{\varphi}}} u_{\ell_{\varphi_*}} \cdot \|T\|_{\ell_{\widetilde{\varphi}}[\ell_{\varphi_*}]}$$

and, by induction, for all $k \in \mathbb{N}$

$$\Lambda_{\ell_\varphi}(T) \le 2^{1/2^{k-1}}(C\log(1+n))^{1/2^k} \cdot (l_{\ell_\varphi} u_{\ell_{\widetilde{\varphi}}} u_{\ell_{\varphi_*}})^{\sum_{i=1}^k (1/2^i)} \cdot \|T\|_{\ell_{\widetilde{\varphi}}[\ell_{\varphi_*}]},$$

which implies (19) with $C = l_{\ell_\varphi} u_{\ell_{\widetilde{\varphi}}} u_{\ell_{\varphi_*}}$. Part (ii) follows immediately from (i) since $\widetilde{\widetilde{\varphi}}$ can be chosen to be φ. ∎

For a power function $\varphi(t) = t^p$, $p \ge 2$ one has $\widetilde{\varphi}(t) = \varphi(t) = t^p$ and $\varphi_*(t) = ct^{p'}$ ($\star$) for some $c > 0$, hence $\ell_\varphi = \ell_{\overline{\varphi}} = \ell_p$ and $\ell_{\varphi_*} = \ell_{p'}$. Since $l_{\ell_p} = u_{\ell_p} = u_{\ell_{p'}} = 1$, we obtain (after deleting any constant caused by the constant in ($\star$) by the product trick) the classical result due to [21] (for $p = 2$) and [12] (for an elementary proof see [3]):

Corollary 13: *Let $2 \le p < \infty$. Then for all n and every $n\times n$ matrix $T = (\tau_{ij})$*

$$\left(\sum_{i=1}^n |\lambda_i(T)|^p\right)^{1/p} \le \left(\sum_{j=1}^n \left(\sum_{i=1}^n |\tau_{ij}|^{p'}\right)^{p/p'}\right)^{1/p}.$$

We conclude with a result for the cotype q case.

THEOREM 14: *For $2 \leq q < \infty$ let φ be an Orlicz function which satisfies $\varphi(\lambda t) \leq K\lambda^q\varphi(t)$ for some $K > 0$ and all $\lambda \geq 1$, $t \geq 0$.*

(i) *If φ is supermultiplicative, then there exists a constant $C > 0$ such that for all n and every $n \times n$ matrix $T = (\tau_{ij})$*

$$\Lambda_{\ell_q}(T) \leq C\|T\|_{\ell_{\widetilde{\varphi}}[\ell_{\varphi_*}]}.$$

(ii) *If φ is submultiplicative, then there exists a constant $C > 0$ such that for all n and every $n \times n$ matrix $T = (\tau_{ij})$*

$$\Lambda_{\ell_q}(T) \leq C\|T\|_{\ell_\varphi[\ell_{(\widetilde{\varphi})_*}]}.$$

Proof: ℓ_φ is of cotype q if and only if φ satisfies the above condition (cf. [14]), and in this case $\widetilde{\varphi}$ is defined. If φ is supermultiplicative, then by [6] it satisfies a lower ℓ_φ-estimate, ℓ_{φ_*} an upper ℓ_{φ_*}-estimate and $\ell_{\widetilde{\varphi}}$ an upper $\ell_{\widetilde{\varphi}}$-estimate. Then (i) follows by (9) and Lemma 11 as in the preceding proof. The proof of (ii) is similar. ∎

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